Preview of coming attractions.

1) Metric spaces
   One of the central notion of Analysis is a notion of **limit**. Informally: x_n is converging to x if for n large enough, x_n is close to x. We will discuss the formal notion of limit in much more detail later on, but let us concentrate of the notion of ***close.*** Where would it make sense to talk about limits, and, hence, continuous functions, etc.? Surely, we can not be too abstract. So, what additional structure do we need to talk about the Calculus concepts?
   To say that two elements of the set are close, we need a notion of distance between two points. And the distance should be nonnegative, symmetric, satisfy the triangle inequality, and of course the distance between two *distinct* points should be *positive*. Turns out that such spaces, so called *metric spaces* are very common, and we can talk about spaces of, say, continuous functions on the interval with max-distance, or even the space of subsets with Hausdorff distance. It is amazing how many analytic concepts work on Metric spaces the same way they worked in Calculus, without any changes whatsoever. And you do not need to create a separate theory for each metric space -- the theory just uses the existence of good distance.

2) Riemann integrability
   You have all encountered the integrals in Calculus -- not as the operation opposite to the differentiation, but as the limit of the Riemannian sums like this:
   $$\int_a^b F(x)dx = \lim \sum_{j=0}^{n} f(y_j)\,(x_{j+1} - x_j)$$
   where one takes the limit when the size of the partition $a = x_0 < x_1 < \cdots < x_{n+1} = b$, i.e. $\max_j(\,x_{j+1} - x_j)$, tends to zero, and $x_j \le y_j \le x_{j+1}$. When does it work? When does the limit exist? In Calculus, it it is explained that it works if f is continuous. But what if f is some very discontinuous function - one encounters them quite often in real life. You surely can integrate, say, this function:

   But how discontinuous can the function be? Can you integrate the function which is equal to *1* at rational points, and *0* in all irrational?
   Turns out, there is precise answer:

   Theorem(Lebesgue).
   A function F is integrable on the interval *[a,b]* iff F the set of points of discontinuity of F has length zero.

   What does it mean for some exotic set to have length zero? We will discuss it!

Preview of coming attraction - continued

Thursday, December 24, 2009
7:21 PM

3) Polynomial approximation

Polynomials, the functions of the form

$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$

are very easy to work with -- you can differentiate them easily, compute the values, etc., etc. So, it seems to be nice to be able to approximate *any* continuous function by polynomials. Turns out, it is always possible, in a certain sense:

Theorem (Weierstrass)

For every continuous function *F* on an interval *[a,b]* and for any *ε>0* one can find a polynomial *P* such that $\max_x |F(x) - P(x)| < \varepsilon$.

This is a very strong Theorem! Especially because the proof is *constructive.* Sometimes one needs to approximate in another sense, like the square distance, or by other types of "good" functions - this Theorem helps there too.